

A PRODUCT FOR PERMUTATION GROUPS AND TOPOLOGICAL GROUPS

SIMON M. SMITH

ABSTRACT. We introduce a new product for permutation groups. It takes as input two permutation groups, M and N , and produces an infinite group $M \boxtimes N$ which carries many of the permutational properties of M . Under mild conditions on M and N the group $M \boxtimes N$ is simple.

As a permutational product, its most significant property is the following: $M \boxtimes N$ is primitive if and only if M is primitive but not regular, and N is transitive. Despite this remarkable similarity with the wreath product in product action, $M \boxtimes N$ and $M \operatorname{Wr} N$ are thoroughly dissimilar.

The product provides a general way to build exotic examples of non-discrete, simple, totally disconnected, locally compact, compactly generated topological groups from discrete groups.

We use this to solve a well-known open problem from topological group theory, by obtaining the first construction of uncountably many pairwise non-isomorphic simple topological groups that are totally disconnected, locally compact, compactly generated and non-discrete. The groups we construct all contain the same compact open subgroup.

To build the product, we describe a group $\mathcal{U}(M, N)$ that acts on an edge-transitive biregular tree T . This group has a natural universal property and is analogous to the iconic universal group construction of M. Burger and S. Mozes for locally finite regular trees.

1. INTRODUCTION

We introduce a new product for permutation groups, which we call the *box product*, and describe some of its striking properties. Our new product is non-associative. It takes two nontrivial permutation groups, $M \leq \operatorname{Sym}(X)$ and $N \leq \operatorname{Sym}(Y)$, which may be finite or infinite, and yields an infinite permutation group $M \boxtimes N$. The group $M \boxtimes N$ is simple under very mild conditions on M and N (see Theorem 16), and it enjoys a universal property (see Theorem 2(iii)).

When N is finite, the group $M \boxtimes N$ inherits permutational properties from M . For example, consider the important permutational property of *primitivity*. Because primitive actions are minimal, products typically do not preserve primitivity. The iconic exception to this is the wreath product in its product action:

- $M \operatorname{Wr} N$ is primitive in its product action on X^Y if and only if M is primitive and not regular on X , and N is transitive and finite.

Because of this property, the wreath product is the primary tool for building new primitive groups from other primitive groups. It is fundamental to the classification of the finite primitive permutation groups (the O’Nan–Scott Theorem). Compare the above with the following astonishing result for the box product:

- $M \boxtimes N$ is primitive in its natural action if and only if M is primitive and not regular on X , and N is transitive (see Theorem 19).

This research was partially supported by an Australian Research Council Discovery Early Career Researcher Award (project number DE130101521).

The groups $M \text{Wr} N$ and $M \boxtimes N$ are thoroughly dissimilar (for example, $S_3 \text{Wr} S_2$ has order 72 and $S_3 \boxtimes S_2$ has order 2^{\aleph_0}).

If M and N are transitive, and M, N and $M \boxtimes N$ are thought of as topological groups under their respective permutation topologies, when N is compact the group $M \boxtimes N$ inherits topological properties from M . Importantly, it does not inherit discreteness. For example, if every point stabiliser in M is compact and N is compact, then every point stabiliser in $M \boxtimes N$ is compact (Theorem 21); if in addition M is compactly generated, then $M \boxtimes N$ is compactly generated (Theorem 23).

To prove the existence of the box product, we construct a group $\mathcal{U}(M, N)$ which acts on an edge-transitive biregular tree T which need not have finite valency. The action of $\mathcal{U}(M, N)$ on T is *locally-(M, N)*; that is, the stabiliser of any vertex v induces either M or N on the neighbours of v . When M and N are transitive, the group $\mathcal{U}(M, N)$ contains an isomorphic copy of every other locally-(M, N) group, and so it is *universal*. This construction is analogous to the iconic universal group construction of Burger and Mozes ([3, Section 3.2]) for locally finite regular trees.

In the final section of the paper, we use the box product to construct an uncountable family \mathcal{F} of examples. The family \mathcal{F} gives an affirmative answer to a well-known open question from topological group theory: *do there exist uncountably many pairwise non-isomorphic simple topological groups that are totally disconnected, locally compact, compactly generated and non-discrete?* This question appears in print in P.-E. Caprace and T. De Medts' paper [4].

George Willis ([21, Problem 4.3]) asked if there exist two non-isomorphic, simple, totally disconnected, locally compact, compactly generated and non-discrete groups that share a common compact open subgroup. Countably many examples of such pairs are known (see [2] and [4]); in \mathcal{F} there are 2^{\aleph_0} such pairs and all share the same compact open subgroup.

In Remark 30 we outline a general method for using the box product to construct exotic examples of non-discrete simple groups that are totally disconnected, locally compact, and compactly generated. For example, one can take M to be a finitely generated simple group (with some additional properties) and N to be finite, and the resulting product $M \boxtimes N$ inherits many of the properties of M but is not discrete. Since the class of finitely generated simple groups is known to contain many unusual groups (see [14, Theorem C], [15, Theorem 28.7] or [16, Theorem 1.1] for example) we see, for the first time, that the class of simple, totally disconnected, locally compact, compactly generated and non-discrete groups is similarly broad.

2. PRELIMINARIES

2.1. Permutation groups. Let V be a non-empty set, and suppose G acts on V . If $x \in V$ and $g \in G$, we denote the image of x under g by gx , thus following the convention that our permutations act from the left. This notation extends naturally to sets, so if $\Phi \subseteq V$ then $g\Phi$ is the image $\{gx : x \in \Phi\}$. The *setwise stabiliser* of Φ is the group $G_{\{\Phi\}} := \{g \in G : g\Phi = \Phi\}$, and the *pointwise stabiliser* is the group $G_{(\Phi)} := \{g \in G : gx = x, \forall x \in \Phi\}$. If Λ consists of a single element x , then the setwise and pointwise stabilisers of Λ coincide; this group is called the *stabiliser* of x and is denoted by G_x . The set $Gx := \{gx : g \in G\}$ denotes the *orbit* of x (under G), and G is said to be *transitive* if V consists of a single orbit. If Φ is an orbit of G , then G induces a subgroup of $\text{Sym}(\Phi)$ which we denote by $G|_{\Phi}$. The orbits of point stabilisers in G are called *suborbits* of G .

The action of G gives rise to a homomorphism from G to the group $\text{Sym}(V)$ of all permutations of V . If this homomorphism is injective, then G is said to be acting

faithfully; when this occurs we will often consider G to be a subgroup of $\text{Sym}(V)$, identifying G with its image in $\text{Sym}(V)$. Two permutation groups $G \leq \text{Sym}(V)$ and $H \leq \text{Sym}(\Delta)$ are *permutation isomorphic* if there exists a bijection $\phi : V \rightarrow \Delta$ such that the map $g \mapsto \phi g \phi^{-1}$ is an isomorphism from G to H .

A transitive group $G \leq \text{Sym}(V)$ is *primitive* on V if the only G -invariant equivalence relations on V are the trivial relation (each element in V is related only to itself) and the universal relation (each element in V is related to every element in V). If G is transitive, then it is primitive if and only if every point-stabiliser G_x is a maximal subgroup of G . A permutation group is *semi-regular* if every point stabiliser is trivial; a *regular* permutation group is transitive and semi-regular.

2.2. Graphs. In this paper a graph Γ consists of a set $V\Gamma$ and a set $E\Gamma$ of two-element subsets of $V\Gamma$. The elements in $V\Gamma$ are called the *vertices* of Γ , and the elements of $E\Gamma$ the *edges*. The graph is *nontrivial* if $E\Gamma$ is non-empty. If two distinct vertices v and w belong to the same edge, they are said to be *adjacent*. An *arc* in Γ is an ordered pair of adjacent vertices, and the set of all arcs is denoted by $A\Gamma$. Thus, our graphs contain no loops or multiple edges, and between any two adjacent edges there are two arcs, one in each direction. We denote the automorphism group of Γ by $\text{Aut}(\Gamma)$.

If $a \in A\Gamma$, we denote by $o(a)$ and $t(a)$ the vertices such that $a = (o(a), t(a))$, and by \bar{a} the arc $(t(a), o(a))$. We will sometimes write the edge $\{o(a), t(a)\}$ as $\{a, \bar{a}\}$. If v is a vertex in Γ , then $A(v) := \{a \in A\Gamma : o(a) = v\}$ denotes those arcs originating from v , while $\bar{A}(v) := \{a \in A\Gamma : t(a) = v\}$ denotes those arcs that terminate at v . We denote the set of vertices adjacent to v by $B(v)$. These notational conventions extend to sets of vertices, so for example if $W \subseteq V\Gamma$ then $A(W) := \{a \in A\Gamma : o(a) \in W\}$.

The *valency* of v is the cardinal $|B(v)|$; if all valencies are finite the graph is *locally finite*. A *path* is a series of distinct vertices $v_0 v_1 \dots v_n$ such that $v_i \in B(v_{i-1})$ for all integers i satisfying $1 \leq i \leq n$; the *length* of this path is n . A path that consists of a single vertex is called *trivial* or sometimes *empty*. Two vertices are connected if there is a path between them, and the *distance* between two connected vertices v, w , which we denote by $d(v, w)$, is the length of the shortest path between them; if two vertices are not connected then their distance is infinite. A graph is *connected* if the distance between any two vertices is finite. A *cycle* is a series of vertices $v_0 v_1 \dots v_n v_0$ such that $n > 1$ and $v_0 v_1 \dots v_n$ and $v_1 \dots v_n v_0$ are nontrivial paths. A *tree* is a connected graph that contains no cycles. In a tree T , there is a unique path between any two vertices $\alpha, \beta \in V\Gamma$ which we denote by $[\alpha, \beta]_T$. If we wish to exclude β , we write $[\alpha, \beta)_T := [\alpha, \beta]_T \setminus \{\beta\}$.

If W is a set of vertices in a graph Γ , we denote by $\Gamma \setminus W$ the subgraph of Γ induced on $V\Gamma \setminus W$. On the other hand, if W is a set of edges of Γ , then $\Gamma \setminus W$ denotes the graph on $V\Gamma$ with edge set $E\Gamma \setminus W$. If Γ is connected, and $\Gamma \setminus \{\alpha\}$ is disconnected for some $\alpha \in V\Gamma$, then Γ is said to have *connectivity one* and α is called a *cut vertex*. The connected subgraphs of Γ that are maximal subject to the condition that they do not have connectivity one are called *lobes*.

If Γ is vertex transitive, has connectivity one and is not a tree, then every vertex is a cut vertex and there is a tree T_Γ , called the *block-cut-vertex tree* of Γ , which determines the structure of Γ . The block-cut-vertex tree is defined as follows. Let L be a set in bijective correspondence with the set of lobes of Γ . Since $\text{Aut } \Gamma$ acts on the set of lobes of Γ , there is an induced action of $\text{Aut } \Gamma$ on L . The vertex set of T_Γ is the union $L \cup V\Gamma$, and $v \in V\Gamma$ and $\ell \in L$ are adjacent in T_Γ if and only if v lies in the lobe associated with ℓ . The action of $\text{Aut } \Gamma$ on $V \cup L$ preserves this relation, and so $\text{Aut } \Gamma$ acts on T_Γ . This action is easily seen to be faithful, and we

will frequently consider $\text{Aut } \Gamma$ to be a subgroup of $\text{Aut } T_\Gamma$. Figure 1 shows a graph with connectivity one and its block-cut-vertex tree.

A *ray* (also called a *half-line*) in Γ is a sequence of distinct vertices $R = \{v_i\}_{i \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$ we have $v_1 v_2 \dots v_n$ is a path in Γ ; thus a ray is a one-way infinite path. The vertex v_1 is often called *root* of R . The *ends* of Γ are equivalence classes of rays: two rays R_1 and R_2 lie in the same *end* if there is a third ray R_3 that contains infinitely many vertices from both R_1 and R_2 . In the special case of an infinite tree T , the ends of T are particularly easy to picture: for a fixed vertex $x \in VT$, each end ϵ contains precisely one ray whose root is x , and we denote this ray by $[x, \epsilon)_T$.

Let m, n be non-zero cardinal numbers. A tree T is *m-regular*, or sometimes *regular*, if the valency of every vertex equals m . There is a natural bipartition of any tree T , in which any pair of vertices whose distance is even lie in the same part of the partition. If all vertices in one part of this bipartition have valency m , and all vertices in the other part have valency n , we say that T is *(m, n)-biregular*. If $a \in AT$ is an arc in T and both connected components of $T \setminus \{a, \bar{a}\}$ are infinite, then each connected component is called a *half-tree* of T . We denote the half-tree of $T \setminus \{a, \bar{a}\}$ containing $o(a)$ by T_a , so the other connected component of $T \setminus \{a, \bar{a}\}$ is $T_{\bar{a}}$.

Given a permutation group $G \leq \text{Sym}(\Omega)$ and distinct elements $\alpha, \beta \in \Omega$, one can construct a graph whose vertex set is Ω and whose edges are the elements of the orbit $\{\alpha, \beta\}^G$. Such a graph is called an *orbital graph* of G .

2.3. Permutation groups as topological groups. For a thorough introduction see [13] and [22]. If V is any non-empty set, then $\text{Sym}(V)$ can be given a natural topology, that of *pointwise convergence*, under which $\text{Sym}(V)$ is a Hausdorff topological group. A basis of neighbourhoods of the identity is given by point stabilisers of finite subsets of V , and so an open set in $\text{Sym}(V)$ is a union of cosets of point stabilisers of finite subsets of V . Under this topology, if $G \leq \text{Sym}(V)$ then a subgroup of G is open in G if and only if it contains the pointwise stabiliser (in G) of some finite subset of V . We will often refer to this topology as the *permutation topology*.

For any finite subset Φ , the pointwise stabiliser $(\text{Sym}(V))_{(\Phi)}$ is both open and closed in $\text{Sym}(V)$, so with this topology $\text{Sym}(V)$ is totally disconnected. Convergence in $\text{Sym}(V)$ is natural: a set $W \subseteq \text{Sym}(V)$ has a limit point $h \in \text{Sym}(V)$ if and only if, for all finite subsets $\Phi \subseteq V$, there exists $g \in W$ distinct from h such that $gh^{-1} \in \text{Sym}(V)_{(\Phi)}$.

Let G be a subgroup of $\text{Sym}(V)$. The group G is closed if and only if some point stabiliser G_α is closed, which holds if and only if all point stabilisers in G are closed. The group G is compact if and only if G is closed and all G -orbits on V are finite. A closed group G is locally compact if and only if $G_{(\Phi)}$ is compact for some finite $\Phi \subseteq V$ (see [8, Lemma 3.1] for example). In particular, if Γ is a connected locally finite graph, then any closed subgroup G of $\text{Aut}(\Gamma)$ will be totally disconnected and locally compact.

A topological space in which every subset is open is called *discrete*, and so the permutation topology is discrete on G if and only if there is a finite subset $\Phi \subseteq V$ such that $G_{(\Phi)}$ is trivial. If the permutation topology is discrete on G we say G is a *discrete permutation group*.

3. GROUPS ACTING ON TREES WITHOUT INVERSION: A UNIVERSAL GROUP

Throughout, X and Y will be disjoint sets, each containing at least two elements, and $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$ will be nontrivial permutation groups. Let T denote the $(|X|, |Y|)$ -biregular tree. Let V_X be the set of vertices of T with valency

$|X|$, and V_Y the set of vertices with valency $|Y|$, so V_X and V_Y are the two parts of the natural bipartition of T . A function $\mathcal{L} : AT \rightarrow X \cup Y$ is called a *legal colouring of X and Y* if it satisfies:

- (i) for all $v \in V_X$, the restriction $\mathcal{L}|_{A(v)} : A(v) \rightarrow X$ is a bijection;
- (ii) for all $v \in V_Y$, the restriction $\mathcal{L}|_{A(v)} : A(v) \rightarrow Y$ is a bijection; and
- (iii) for all $v \in VT$, the image of $\mathcal{L}|_{\overline{A(v)}}$ contains precisely one element.

One may easily verify that it is always possible to construct a legal colouring of X and Y .

If $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$, and \mathcal{L} is a legal colouring of X and Y , define

$$\mathcal{U}_{\mathcal{L}}(M, N) := \left\{ g \in (\text{Aut } T)_{\{V_X\}} : \mathcal{L}|_{A(gv)} g|_{A(v)} \mathcal{L}|_{A(v)}^{-1} \in \begin{cases} M & \text{for all } v \in V_X \\ N & \text{for all } v \in V_Y \end{cases} \right\}.$$

Verifying that $\mathcal{U}_{\mathcal{L}}(M, N)$ is a subgroup of $\text{Aut } T$ is tedious but not difficult.

The first construction of the group $\mathcal{U}_{\mathcal{L}}(M, N)$ was inspired by [19, Section 2.2], and $\mathcal{U}_{\mathcal{L}}(M, N)$ can be constructed using refinements of the arguments in [19] (which use relational structures), so long as M and N are closed (in their respective permutation topologies). It acts on a biregular tree that may have infinite valency, and is analogous to Burger and Mozes' universal group $U(F)$ ([3, Section 3.2]), which acts on a locally finite regular tree.

Since $\mathcal{U}_{\mathcal{L}}(M, N)$ preserves the parts V_X and V_Y , it induces a subgroup of $\text{Sym}(V_Y)$. This motivates the following definition.

Definition 1. The *box product of M and N* , denoted $M \boxtimes_{\mathcal{L}} N$, is the subgroup of $\text{Sym}(V_Y)$ that is induced by $\mathcal{U}_{\mathcal{L}}(M, N)$. As we shall see in Proposition 5, for two legal colourings $\mathcal{L}, \mathcal{L}'$ the groups $M \boxtimes_{\mathcal{L}} N$ and $M \boxtimes_{\mathcal{L}'} N$ are permutationally isomorphic, and so we write $M \boxtimes N$ instead of $M \boxtimes_{\mathcal{L}} N$ when there is no chance of ambiguity.

Let us say that a group $H \leq \text{Aut } T$ is *locally-(M, N)* if H fixes setwise the parts V_X and V_Y setwise, and for all vertices v of T the group $H_v|_{B(v)} \leq \text{Sym}(B(v))$ induced by the vertex stabiliser H_v is permutation isomorphic to M if $v \in V_X$ and N if $v \in V_Y$. As a subgroup of $\text{Aut } T$, the group $\mathcal{U}_{\mathcal{L}}(M, N)$ has the following properties.

Theorem 2. Suppose X, Y are disjoint sets of cardinality at least two, and T is the $(|X|, |Y|)$ -biregular tree. Given permutation groups $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$, and a legal colouring \mathcal{L} of T ,

- (i) $\Delta \subseteq X \cup Y$ is an orbit of M or N if and only if $t(\mathcal{L}^{-1}\Delta)$ is an orbit of $\mathcal{U}_{\mathcal{L}}(M, N)$;
- (ii) given legal colourings $\mathcal{L}, \mathcal{L}'$, the groups $\mathcal{U}_{\mathcal{L}}(M, N)$ and $\mathcal{U}_{\mathcal{L}'}(M, N)$ are conjugate in $\text{Aut } T$;
- (iii) if M and N are transitive, and $H \leq \text{Aut } T$ is locally-(M, N), then for some legal colouring \mathcal{L} we have

$$H \leq \mathcal{U}_{\mathcal{L}}(M, N);$$

- (iv) $\mathcal{U}_{\mathcal{L}}(M, N)$ is locally-(M, N);
- (v) if M and N are closed, then $\mathcal{U}_{\mathcal{L}}(M, N)$ is a closed subgroup of $\text{Aut } T$.

Thus, when M and N are transitive, $\mathcal{U}_{\mathcal{L}}(M, N)$ is the universal locally-(M, N) group, in that it contains a permutationally isomorphic copy of every locally-(M, N) group acting on the biregular tree T . Theorem 2 follows from Lemma 3, Propositions 4–6 and Lemmas 7–8.

Our arguments in this section rely on manipulating sequences of bijections and restrictions of functions. Although these manipulations can look rather daunting, they are in fact very simple and involve only the following basic steps. If $g, h \in (\text{Aut } T)_{\{V_X\}}$, then for all $w \in VT$,

- (i) $g|_{A(w)}^{-1} = g^{-1}|_{A(gw)}$;
- (ii) $g|_{A(hw)}h|_{A(w)}g^{-1}|_{A(gw)} = (ghg^{-1})|_{A(gw)}$; and
- (iii) if $\mathcal{L} = \mathcal{L}'g$, then $\mathcal{L}|_{A(w)} = \mathcal{L}'|_{A(gw)}g|_{A(w)}$ and $\mathcal{L}|_{A(w)}^{-1} = g^{-1}|_{A(gw)}\mathcal{L}'|_{A(gw)}^{-1}$.

In what follows, \mathcal{L} and \mathcal{L}' are arbitrary legal colourings of X and Y . To simplify notation, we associate each constant function $\mathcal{L}|_{\overline{A(w)}}$ with its image in $X \cup Y$, so for all $w \in VT$ we have $\mathcal{L}|_{\overline{A(w)}} \in X \cup Y$.

We begin with a lemma that underpins most results in this section.

Lemma 3. *If the distance between $v, v' \in VT$ is even, then there exists $\sigma \in \text{Sym}(X \cup Y)_{\{X\}}$ satisfying $\sigma\mathcal{L}'|_{\overline{A(v')}} = \mathcal{L}|_{\overline{A(v)}}$, and for all such σ there exists a unique automorphism $g \in (\text{Aut } T)_{\{V_X\}}$ such that $gv = v'$ and $\mathcal{L} = \sigma\mathcal{L}'g$.*

Proof. Since \mathcal{L} and \mathcal{L}' are legal colourings, we can always find $\sigma \in \text{Sym}(X \cup Y)_{\{X\}}$ satisfying $\sigma\mathcal{L}'|_{\overline{A(v')}} = \mathcal{L}|_{\overline{A(v)}}$. For each integer $n \geq 0$, let B_n (resp. B'_n) be the subtree of T induced by those vertices whose distance in T from v (resp. v') is at most n . Because $d(v, v')$ is even, v and v' belong to the same part of the bipartition of T .

Let $g_0 : B_0 \rightarrow B'_0$ be the map taking v to v' . Our proof that an appropriate element $g \in \text{Aut } T$ exists relies on an easy induction argument with the following induction hypothesis: for all integers $n \geq 1$, there exists a graph isomorphism $g_n : B_n \rightarrow B'_n$ such that $\sigma\mathcal{L}'|_{B'_n}g_n = \mathcal{L}|_{B_n}$ and $g_n|_{B_{n-1}} = g_{n-1}$. The base case is true because $\mathcal{L}'|_{A(v')}^{-1}\sigma^{-1}\mathcal{L}|_{A(v)}$ is a bijection from $A(v)$ to $A(v')$, and so it induces a graph isomorphism $g_1 : B_1 \rightarrow B'_1$ with $g_1v = v'$.

Suppose, for some $n \geq 1$, that our induction hypothesis holds. If w is a vertex in $B_n \setminus B_{n-1}$, then there is a unique arc a in B_n whose origin is w , and g_na is the unique arc in B'_n whose origin is g_nw . Now $\mathcal{L}'|_{A(g_nw)}^{-1}\sigma^{-1}\mathcal{L}|_{A(w)}$ maps a to g_na and so it induces a bijection $h_w : B(w) \setminus VB_n \rightarrow B(g_nw) \setminus VB'_n$. Let g_{n+1} be g_n extended by all these h_w , for $w \in B_n \setminus B_{n-1}$, and note that g_{n+1} is a graph isomorphism from B_{n+1} to B'_{n+1} with $g_{n+1}|_{B_n} = g_n$. It remains to show that $\sigma\mathcal{L}'|_{B'_{n+1}}g_{n+1} = \mathcal{L}|_{B_{n+1}}$. We need only consider arcs in $B_{n+1} \setminus B_n$. If a is such an arc, then either $t(a) \in VB_n$ or $o(a) \in VB_n$. If $t(a) \in VB_n$, chose an arc b in B_n with $t(b) = t(a)$. Since $t(b) = t(a)$ and $t(g_{n+1}a) = t(g_{n+1}b)$ we have $\mathcal{L}b = \mathcal{L}a$ and $\mathcal{L}'g_{n+1}b = \mathcal{L}'g_{n+1}a$. Moreover, $\sigma\mathcal{L}'g_{n+1}b = \mathcal{L}b$ by the induction hypothesis, and so $\mathcal{L}a = \mathcal{L}b = \sigma\mathcal{L}'g_{n+1}b = \sigma\mathcal{L}'g_{n+1}a$. On the other hand, if $w := o(a) \in VB_n$ then $g_{n+1}w = g_nw$ and $g_{n+1}t(a) = h_w t(a)$. Hence (by definition of h_w) we have $\sigma\mathcal{L}'g_{n+1}a = \mathcal{L}a$. Thus, our induction hypothesis is true for all integers $n \geq 1$.

We now construct an element $g \in (\text{Aut } T)_{\{V_X\}}$ that satisfies $\sigma\mathcal{L}'g = \mathcal{L}$ and $gv = v'$. For each vertex $w \in VT \setminus \{v\}$ there is a unique $n(w) \in \mathbb{N}$ such that $w \in B_{n(w)} \setminus B_{n(w)-1}$. Define $g : VT \rightarrow VT$ to be such that $gv := v'$ and for all $w \in VT \setminus \{v\}$ set $gx := g_{n(w)}x$. One may easily verify that $g \in (\text{Aut } T)_{\{V_X\}}$. Moreover, for all $a = (w, w') \in AT$, there exists an m such that $a \in AB_m$. Then $ga = (g_{n(w)}w, g_{n(w')}w') = (g_mw, g_mw') = g_ma$. Hence $\sigma\mathcal{L}'ga = \sigma\mathcal{L}'g_ma = \mathcal{L}a$.

It remains to prove that g is unique. Suppose $h \in \text{Aut } T$ with $\sigma\mathcal{L}'h = \mathcal{L}$ and $hv = v'$. We claim that if $gh^{-1}w = w$ for some $w \in VT$, then gh^{-1} fixes $B(w)$ pointwise. Indeed, if $a \in A(w)$ then $gh^{-1}a \in A(w)$, and $\sigma\mathcal{L}'gh^{-1}a = \mathcal{L}h^{-1}a =$

$\sigma\mathcal{L}'hh^{-1}a = \sigma\mathcal{L}'a$. Our claim follows immediately because $\mathcal{L}'|_{A(w)}$ is a bijection. Since $hv = v'$, we have $gh^{-1}v' = v'$, and our claim tells us that gh^{-1} fixes $B(v')$ pointwise. Since T is connected, gh^{-1} fixes VT pointwise, and is therefore the identity. \square

The orbits of $\mathcal{U}_{\mathcal{L}}(M, N)$ relate naturally to those of M and N .

Proposition 4. *Vertices v, v' in V_X (resp. V_Y) lie in the same orbit of $\mathcal{U}_{\mathcal{L}}(M, N)$ if and only if $\mathcal{L}|_{\overline{A}(v)}$ and $\mathcal{L}|_{\overline{A}(v')}$ lie in the same orbit of N (resp. M).*

Proof. Suppose $v, v' \in V_X$ (a symmetric argument holds for $v, v' \in V_Y$). If there exists $\sigma \in N$ such that $\sigma\mathcal{L}|_{\overline{A}(v)} = \mathcal{L}|_{\overline{A}(v')}$, let $\hat{\sigma}$ be the element in $\text{Sym}(X \cup Y)_{\{Y\}}$ that is σ on Y and trivial on X . By Lemma 3, there exists a unique $g \in (\text{Aut } T)_{\{V_X\}}$ such that $gv = v'$ and $\mathcal{L} = \hat{\sigma}\mathcal{L}g$. If $w \in V_X$, then $\mathcal{L}|_{A(gw)g|_{A(w)}\mathcal{L}|_{A(w)}^{-1}} = (\mathcal{L}g)|_{A(w)}\mathcal{L}|_{A(w)}^{-1} = (\hat{\sigma}^{-1}\mathcal{L})|_{A(w)}\mathcal{L}|_{A(w)}^{-1} = \hat{\sigma}^{-1}|_X \in M$, and if $w \in V_Y$ then a similar argument shows $\mathcal{L}|_{A(gw)g|_{A(w)}\mathcal{L}|_{A(w)}^{-1}} \in N$. Hence $g \in \mathcal{U}_{\mathcal{L}}(M, N)$.

Conversely, if $gv = v'$ for some $g \in \mathcal{U}_{\mathcal{L}}(M, N)$, fix $(w, v) \in \overline{A}(v)$ and define $\sigma := \mathcal{L}|_{A(gw)g|_{A(w)}\mathcal{L}|_{A(w)}^{-1}} \in N$. Then $\sigma\mathcal{L}(w, v) = \mathcal{L}(gw, v')$. By the definition of \mathcal{L} , we have $\mathcal{L}|_{\overline{A}(v)} = \mathcal{L}(w, v)$ and $\mathcal{L}|_{\overline{A}(v')} = \mathcal{L}(gw, v')$. Hence $\sigma\mathcal{L}|_{\overline{A}(v)} = \mathcal{L}|_{\overline{A}(v')}$. \square

Any two legal colourings give rise to essentially the same permutation group.

Proposition 5. *The groups $\mathcal{U}_{\mathcal{L}}(M, N)$ and $\mathcal{U}_{\mathcal{L}'}(M, N)$ are conjugate in $\text{Aut } T$.*

Proof. Choose vertices $v, v' \in VT$ such that $\mathcal{L}|_{\overline{A}(v)}$ and $\mathcal{L}'|_{\overline{A}(v')}$ are equal. By Lemma 3, there is a unique element $g \in (\text{Aut } T)_{\{V_X\}}$ such that $gv = v'$ and $\mathcal{L} = \mathcal{L}'g$. Choose $h \in (\text{Aut } T)_{\{V_X\}}$, and set $h' := ghg^{-1}$. If $w' \in VT$, then $w := g^{-1}w'$ and w' must both lie in V_X , or both lie in V_Y , and

$$\begin{aligned} \mathcal{L}'|_{A(h'w')}h'|_{A(w')}\mathcal{L}'|_{A(w')}^{-1} &= \mathcal{L}'|_{A(ghw)g|_{A(hw)}h|_{A(w)}g^{-1}|_{A(gw)}\mathcal{L}'|_{A(gw)}^{-1} \\ &= \mathcal{L}|_{A(hw)}h|_{A(w)}\mathcal{L}|_{A(w)}^{-1}. \end{aligned}$$

Hence $h' \in \mathcal{U}_{\mathcal{L}'}(M, N)$ if and only if $h \in \mathcal{U}_{\mathcal{L}}(M, N)$. \square

If M and N are transitive, then $\mathcal{U}_{\mathcal{L}}(M, N)$ contains an isomorphic copy of every locally- (M, N) group.

Proposition 6. *If M and N are transitive, and $H \leq \text{Aut } T$ is locally- (M, N) , then $H \leq \mathcal{U}_{\mathcal{L}}(M, N)$ for some legal colouring \mathcal{L} of T .*

Proof. Suppose M and N are transitive, and $H \leq \text{Aut } T$ is locally- (M, N) , so H has precisely two orbits on VT , acting transitively on V_X and V_Y . Fix adjacent vertices $p \in V_X$ and $q \in V_Y$. Since $H_p|_{B(p)}$ is permutation isomorphic to M there exists a bijection $\phi : B(p) \rightarrow X$ such that $\phi H_p|_{B(p)}\phi^{-1} = M$. Similarly, there exists a bijection $\psi : B(q) \rightarrow Y$ such that $\psi H_q|_{B(q)}\psi^{-1} = N$. For each integer $n \geq 0$, let S_n be the set of vertices in T whose distance from p is precisely n . For $v \in \bigcup_{i \geq 2} S_i$, there is a unique path $[p, v]_T$ in T between p and v ; denote the unique vertex in $B(v) \cap [p, v]_T$ by v_1 and the unique vertex in $B(v_1) \cap [p, v_1]_T$ by v_2 .

We inductively construct a subset $J := \{h_v : v \in VT\}$ of H that we will use to define a legal colouring \mathcal{L} such that $H \leq \mathcal{U}_{\mathcal{L}}(M, N)$. Each element $h_v \in J$ will have the property that $h_v p = v$ if $v \in V_X$ and $h_v q = v$ if $v \in V_Y$.

Let h_p be the identity, and for each $v \in S_1$, choose $h_v \in H_p$ such that $h_v q = v$. Notice that for all $v \in \bigcup_{i=0}^1 S_i$, $h_v p = v$ if $v \in V_X$ and $h_v q = v$ if $v \in V_Y$. Let $n \in \mathbb{N}$, and suppose, for each $v \in \bigcup_{i=0}^n S_i$, we have chosen an element $h_v \in H$ such that

$h_v p = v$ if $v \in V_X$ and $h_v q = v$ if $v \in V_Y$. For $v \in S_{n+1}$, choose $h \in H_{v_1}$ such that $h v_2 = v$ (which we can do because H_{v_1} is transitive on $B(v_1)$) and set $h_v := h h_{v_2}$. If $v \in V_X$, then $v_2 \in V_X$ and $h_v p = v$, and similarly if $v \in V_Y$ then $h_v q = v$. Note that for all $v \in \bigcup_{i \geq 2} S_i$ we have $h_{v_2} h_v^{-1} = h \in H_{v_1}$, and for all $v, w \in S_1$ we have $h_w, h_v \in H_p$, so $h_w^{-1} h_v \in H_p$.

We now describe a colouring \mathcal{L} , and show that it is a legal colouring. Define $\mathcal{L} : AT \rightarrow X \cup Y$ as follows: for $v \in V_X$ let $\mathcal{L}|_{A(v)} : A(v) \rightarrow X$ be $a \mapsto \phi h_v^{-1} t(a)$, and for $v \in V_Y$ let $\mathcal{L}|_{A(v)} : A(v) \rightarrow Y$ be $a \mapsto \psi h_v^{-1} t(a)$. It is clear that for all $v \in VT$ the map $\mathcal{L}|_{A(v)}$ is a bijection, so it remains to show that condition (iii) of the definition holds; that is, the image of $\mathcal{L}|_{\bar{A}(v)}$ has cardinality one for all $v \in VT$. We show this first for p , then for all $v \in S_1$, and finally for all $v \in \bigcup_{n \geq 2} S_n$.

If (w, p) is any arc in $\bar{A}(p)$, then $w \in V_Y$ and $\mathcal{L}|_{\bar{A}(p)}(w, p) = \mathcal{L}|_{A(w)}(w, p) = \psi h_w^{-1} p = \psi p$ (because $w \in S_1$ so $h_w \in H_p$). Therefore the cardinality of the image of $\mathcal{L}|_{\bar{A}(p)}$ is one.

If $v \in S_1$ and $(w, v) \in \bar{A}(v)$, then either $w = p$, or $w \in S_2 \subseteq V_X$. If $w = p$ then $\mathcal{L}|_{\bar{A}(v)}(w, v) = \phi v$. On the other hand, if $w \in S_2$ then $w_1 = v$ and $w_2 = p$. Hence $\mathcal{L}|_{\bar{A}(v)}(w, v) = \mathcal{L}|_{A(w)}(w, v) = \phi h_w^{-1} v$ and $h_w \in H_{w_1} h_{w_2} = H_v h_p = H_v$. Thus $\mathcal{L}|_{\bar{A}(v)}(w, v) = \phi v$. Therefore, if $v \in S_1$ then the cardinality of the image of $\mathcal{L}|_{\bar{A}(v)}$ is always one.

Suppose that $v \in S_n$, for some $n \geq 2$. Choose $(w, v) \in \bar{A}(v)$ such that $w \neq v_1$; then $v = w_1$ and $v_1 = w_2$. If $v \in V_X$, then $w \in V_Y$ and $\mathcal{L}|_{\bar{A}(v)}(w, v) = \mathcal{L}|_{A(w)}(w, v) = \psi h_w^{-1} v = \psi h_w^{-1} w_1$. Since $h_w^{-1} \in h_{w_2}^{-1} H_{w_1}$, we have $\mathcal{L}|_{\bar{A}(v)}(w, v) = \psi h_{w_2}^{-1} w_1 = \psi h_{v_1}^{-1} v = \mathcal{L}|_{\bar{A}(v)}(v_1, v)$. Hence, the cardinality of the image of $\mathcal{L}|_{\bar{A}(v)}$ is one. A symmetric argument shows that the same is true if v is instead chosen from V_Y . We have thus demonstrated that \mathcal{L} is a legal colouring.

Finally, we show $H \leq \mathcal{U}_{\mathcal{L}}(M, N)$. Choose $g \in H$ and $v \in VT$. If $v \in V_X$ then $gv \in V_X$ and $h_{gv}^{-1} g h_v \in H_p$. Hence $\mathcal{L}|_{A(gv)} g|_{A(v)} \mathcal{L}|_{A(v)}^{-1} \in \phi H_p|_{B(p)} \phi^{-1} = M$. Similarly, if $v \in V_Y$ then $gv \in V_Y$ and $h_{gv}^{-1} g h_v \in H_q$, so $\mathcal{L}|_{A(gv)} g|_{A(v)} \mathcal{L}|_{A(v)}^{-1} \in \psi H_q|_{B(q)} \psi^{-1} = N$. Therefore $g \in \mathcal{U}_{\mathcal{L}}(M, N)$. \square

In the proposition above, the requirement that M and N be transitive is essential. For example, if G is the automorphism group of the graph Γ pictured in Figure 1, then G is transitive on the vertices of Γ and induces a faithful action on its block-cut-vertex tree T_Γ . In its action on T_Γ , it is easily seen that the group G has two orbits: one orbit consists of the T_Γ -vertices corresponding to the vertices of Γ , and the other consists of those T_Γ -vertices that correspond to the lobes of Γ . Choose any pair v and w of adjacent vertices in T_Γ , and let M denote $G_v|_{B(v)}$ and N denote $G_w|_{B(w)}$. If it were possible to find a legal colouring \mathcal{L} of T_Γ such that $G \leq \mathcal{U}_{\mathcal{L}}(M, N)$, then this would contradict Proposition 4, because neither M nor N are transitive.

Lemma 7. $\mathcal{U}_{\mathcal{L}}(M, N)$ is locally- (M, N) .

Proof. Fix $v \in VT$. If $v \in V_X$ (resp. $v \in V_Y$) we claim the map $g|_{A(v)} \mapsto \mathcal{L}|_{A(v)} g|_{A(v)} \mathcal{L}|_{A(v)}^{-1}$ is an isomorphism from $(\mathcal{U}_{\mathcal{L}}(M, N))_v|_{A(v)}$ to M (resp. N). We prove this only for $v \in V_X$ since a similar argument works for $v \in V_Y$.

It is clear this map is a monomorphism. Fix $\sigma \in M$, and let $\hat{\sigma} \in \text{Sym}(X \cup Y)$ be the permutation that equals σ on X and is the identity on Y . Now $\hat{\sigma} \mathcal{L}$ is

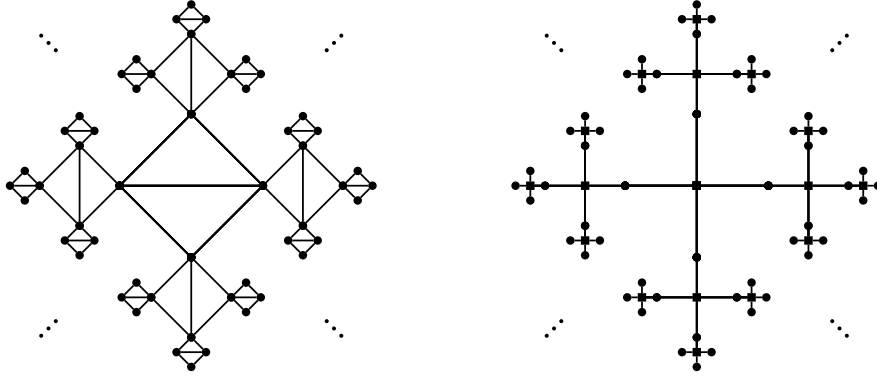


FIGURE 1. An infinite vertex-transitive graph Γ (left) and its block-cut-vertex tree T_Γ (right)

a legal colouring, and Lemma 3 guarantees that there exists a unique element $g \in (\text{Aut } T)_{\{V_X\}}$ such that $gv = v$ and $\mathcal{L}g = \hat{\sigma}\mathcal{L}$. Hence, for all $w \in VT$ we have $\mathcal{L}|_{A(gw)}g|_{A(w)}\mathcal{L}|_{A(w)}^{-1} = \hat{\sigma}|_{\mathcal{L}(A(w))}$, and from this it follows that $g \in (\mathcal{U}_{\mathcal{L}}(M, N))_v$ and $\mathcal{L}|_{A(v)}g|_{A(v)}\mathcal{L}|_{A(v)}^{-1} = \sigma$. Finally, we observe that the permutation groups induced by $(\mathcal{U}_{\mathcal{L}}(M, N))_v$ on $A(v)$ and $B(v)$ are permutation isomorphic. \square

Lemma 8. *If M is a closed subgroup of $\text{Sym}(X)$ and N is a closed subgroup of $\text{Sym}(Y)$, then $\mathcal{U}_{\mathcal{L}}(M, N)$ is a closed subgroup of $\text{Aut } T$.*

Proof. One may easily verify that $A := (\text{Aut } T)_{\{V_X\}}$ is a closed subgroup of $\text{Aut } T$. Let $G := \mathcal{U}_{\mathcal{L}}(M, N)$. We show that $A \setminus G$ is open. Fix $g \in A \setminus G$. By the definition of G , we may choose some vertex $v \in V_X$ such that $\mathcal{L}|_{A(gv)}g|_{A(v)}\mathcal{L}|_{A(v)}^{-1} \notin M$, or some $v \in V_Y$ such that $\mathcal{L}|_{A(gv)}g|_{A(v)}\mathcal{L}|_{A(v)}^{-1} \notin N$. Suppose v is such a vertex, and for all $h \in A$ write $\bar{h} := \mathcal{L}|_{A(hv)}h|_{A(v)}\mathcal{L}|_{A(v)}^{-1}$.

We claim if that $v \in V_X$ and $\bar{g} \notin M$, then there exists an open subset of $A \setminus G$ that contains g . Indeed, suppose $v \in V_X$, and note that $\bar{g} \notin M$. Since M is a closed subgroup of $\text{Sym}(X)$, and $\bar{g} \in \text{Sym}(X)$, there exists an open subset $U \subseteq \text{Sym}(X) \setminus M$ containing \bar{g} . Now $\bar{g}^{-1}U$ is a neighbourhood of the identity, and so it contains the pointwise stabiliser (in $\text{Sym}(X)$) of some finite subset $F \subseteq X$. Hence $\bar{g} \in \bar{g}\text{Sym}(X)_{(F)} \subseteq U \subseteq \text{Sym}(X) \setminus M$. Let Φ denote the finite set $\{v\} \cup \{w \in B(v) : \mathcal{L}|_{A(v)}(v, w) \in F\}$. To prove our claim, it is sufficient to show that the open set $gA_{(\Phi)}$ is contained in $A \setminus G$. If $h \in gA_{(\Phi)}$, then $\bar{h}x = \bar{g}x$ for all $x \in F$. Therefore $\bar{h} \in \bar{g}\text{Sym}(X)_{(F)}$ and so $\bar{h} \notin M$, and from this we may conclude that $h \notin G$. Thus $gA_{(\Phi)} \subseteq A \setminus G$ and our claim is true.

If one interchanges X and M with Y and N in the above argument, then we obtain a proof that if $v \in V_Y$, then there exists an open subset of $A \setminus G$ that contains g . Hence, every element in $A \setminus G$ lies in an open subset of A that is disjoint from G , so $A \setminus G$ is open and G is closed. \square

We next prove that $\mathcal{U}_{\mathcal{L}}(M, N)$ has a subgroup isomorphic to M and a subgroup isomorphic to N . These subgroups are contained in point stabilisers in $\mathcal{U}_{\mathcal{L}}(M, N)$.

If we are given $\mu \in M$, let $\hat{\mu} \in \text{Sym}(X \cup Y)$ be such that $\hat{\mu}|_X = \mu$ and $\hat{\mu}|_Y = 1_N$. By Lemma 3, for each vertex $v \in V_X$ there is a unique automorphism $g_{\mu, v} \in (\text{Aut } T)_{V_X}$ such that $g_{\mu, v}v = v$ and $\mathcal{L} = \hat{\mu}\mathcal{L}g_{\mu, v}$. One may quickly verify that $g_{\mu, v} \in \mathcal{U}_{\mathcal{L}}(M, N)$. For $v \in V_Y$ and $\mu \in N$ define $g_{\mu, v}$ similarly.

Proposition 9. *If $v \in V_X$ and $w \in V_Y$, then $\hat{M}(v) := \{g_{\mu,v} : \mu \in M\}$ is a subgroup of $\mathcal{U}_{\mathcal{L}}(M, N)_v$ and $\hat{N}(w) := \{g_{\tau,w} : \tau \in N\}$ is a subgroup of $\mathcal{U}_{\mathcal{L}}(M, N)_w$. Moreover, $\hat{M}(v)$ is isomorphic to M , and $\hat{N}(w)$ is isomorphic to N .*

Proof. Suppose $v \in V_X$. If $\mu, \tau \in M$, then $g_{\mu,v}g_{\tau,v}v = v = g_{\tau\mu,v}v$ and $\hat{\tau}\hat{\mu}\mathcal{L}g_{\tau\mu,v} = \mathcal{L} = \hat{\tau}(\hat{\mu}\mathcal{L}g_{\mu,v})g_{\tau,v}$. Since $\hat{\tau}\hat{\mu}$ fixes Y pointwise, we have $\hat{\tau}\hat{\mu}\mathcal{L}|_{\bar{A}(v)} = \mathcal{L}|_{\bar{A}(v)}$. It follows immediately from Lemma 3 that $g_{\tau\mu,v} = g_{\mu,v}g_{\tau,v}$. Similarly, it follows from Lemma 3 that if e denotes the identity in M , then $g_{e,v} = 1 \in \mathcal{U}_{\mathcal{L}}(M, N)$. Finally, we have $g_{\mu,v}g_{\mu^{-1},v} = g_{\mu^{-1}\mu,v} = g_{e,v} = 1 \in \mathcal{U}_{\mathcal{L}}(M, N)$, so $g_{\mu^{-1},v} = g_{\mu,v}^{-1}$. Hence $\hat{M}(v) \leq \mathcal{U}_{\mathcal{L}}(M, N)_v$. A similar argument holds for $w \in V_Y$ and $\hat{N}(w)$.

Since $\mu, \tau \in M$, we have $g_{\mu,v} = g_{\tau,v}$ if and only if $\mu = \tau$. Indeed, if $\mu = \tau$, then $\hat{\tau}\mathcal{L}g_{\tau,v} = \mathcal{L} = \hat{\mu}\mathcal{L}g_{\mu,v} = \hat{\tau}\mathcal{L}g_{\mu,v}$, and so $g_{\tau,v} = g_{\mu,v}$ by Lemma 3. On the other hand, if $g_{\tau,v} = g_{\mu,v}$ then $1 = g_{\mu,v}g_{\tau,v}^{-1} = g_{\tau^{-1}\mu,v}$. Hence $\mathcal{L} = \hat{\tau}^{-1}\hat{\mu}\mathcal{L}g_{\tau^{-1}\mu,v} = \hat{\tau}^{-1}\hat{\mu}\mathcal{L}$, so τ and μ must be equal.

Let $\varphi : \hat{M}(v) \rightarrow M$ be given by $g_{\mu,v} \mapsto \mu^{-1}$. This map is obviously surjective, and we have shown it to be well-defined and injective. For $\mu, \tau \in M$ we have $g_{\mu,v}g_{\tau,v} = g_{\tau\mu,v}$, so φ is a homomorphism. A symmetric argument shows that $\hat{N}(w)$ is isomorphic to N . \square

We conclude this section with two lemma, presented without proof, which will be used in the following sections. The first lemma is an immediate consequence of Proposition 4 and Lemma 7.

Lemma 10. *Suppose $v_1, v_2 \in V_X$ and $w_1, w_2 \in V_Y$. Edges $\{v_1, w_1\}$ and $\{v_2, w_2\}$ in T lie in the same orbit of $\mathcal{U}_{\mathcal{L}}(M, N)$ if and only if v_1, v_2 lie in the same orbit of $\mathcal{U}_{\mathcal{L}}(M, N)$ and w_1, w_2 lie in the same orbit of $\mathcal{U}_{\mathcal{L}}(M, N)$. \square*

Recall that T_a and $T_{\bar{a}}$ are the two half-trees of $T \setminus \{a, \bar{a}\}$. To prove the following lemma, simply check that the obvious choice for $g \in \text{Aut } T$ lies in $\mathcal{U}_{\mathcal{L}}(M, N)$.

Lemma 11. *If a is any arc in T , then for all $h \in \mathcal{U}_{\mathcal{L}}(M, N)_a$ there exists $g \in \mathcal{U}_{\mathcal{L}}(M, N)$ such that g fixes $T_{\bar{a}}$ pointwise and $g|_{T_a} = h|_{T_a}$. \square*

4. SIMPLICITY

In his influential paper [20], Jacques Tits introduced the following *property (P)*, which is sometimes known as *Tits' independence property*. Let G act on a (not necessarily locally-finite) tree T . If \mathcal{P} is a non-empty finite or infinite path in T , for each vertex v in T there is a unique vertex $\pi_{\mathcal{P}}(v)$ in \mathcal{P} that is closest to v . This gives rise to a well-defined map on VT , in which each vertex v is mapped to $\pi_{\mathcal{P}}(v)$. For each vertex q in \mathcal{P} , the set $\pi_{\mathcal{P}}^{-1}(q)$ of vertices in T that are mapped to q by $\pi_{\mathcal{P}}$ is the vertex set of a subtree of T . The pointwise stabiliser $G_{(\mathcal{P})}$ of \mathcal{P} leaves each of these subtrees invariant, and so we can define $G_{(\mathcal{P})}^q$ to be the subgroup of $\text{Sym}(\pi_{\mathcal{P}}^{-1}(q))$ induced by $G_{(\mathcal{P})}$. Thus, we have homomorphisms $\varphi_q : G_{(\mathcal{P})} \rightarrow G_{(\mathcal{P})}^q$ for each $q \in V\mathcal{P}$ from which we obtain the natural homomorphism,

$$\varphi : G_{(\mathcal{P})} \rightarrow \prod_{q \in V\mathcal{P}} G_{(\mathcal{P})}^q. \quad (1)$$

The group G is said to have property (P) if the homomorphism (1) is an isomorphism for every possible choice of \mathcal{P} . Intuitively, property (P) means that $G_{(\mathcal{P})}$ acts independently on each of the subtrees branching from \mathcal{P} .

Theorem 12 ([20, Théorème 4.5]). *Suppose T is a tree. If $G \leq \text{Aut } T$ satisfies property (P), no proper non-empty subtree of T is invariant under G and no end of T is fixed by G , then the group $G^+ := \langle G_{(v,w)} : \{v, w\} \in ET \rangle$ is simple (and possibly trivial). \square*

Theorem 13. *Let X and Y be finite or infinite sets whose cardinality is at least two. Suppose $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$ are two permutation groups, and let T be the $(|X|, |Y|)$ -biregular tree. If \mathcal{L} is a legal colouring of X and Y , then $\mathcal{U}_{\mathcal{L}}(M, N)$ satisfies Tits' independence property (P).*

Proof. Let \mathcal{P} be any non-empty path in T and let G denote $\mathcal{U}_{\mathcal{L}}(M, N)$. Suppose we are given $g \in G_{(\mathcal{P})}$ and $p \in V\mathcal{P}$. Note that $B(p)$ contains either one or two vertices in \mathcal{P} , so we may write $B(p) \cap V\mathcal{P} = \{q, q'\}$, where q, q' could be equal. By Lemma 11 there exists $g' \in G$ such that g' fixes $T_{(q,p)}$ pointwise and agrees with g on $T_{(p,q)}$. The element g' fixes \mathcal{P} pointwise, and so in particular it lies in $G_{(p,q')}$, and so we may apply the lemma again to deduce the existence of $g_p \in G$ such that g_p fixes $T_{(q',p)}$ pointwise and agrees with g' on $T_{(p,q')}$. The element g_p therefore fixes $VT \setminus \pi^{-1}(p)$ pointwise, agrees with g on $\pi^{-1}(p)$, and satisfies $g_p|_{\pi^{-1}(p)} \in G_{(\mathcal{P})}^p$.

Now $g = \prod_{p \in V\mathcal{P}} g_p$, and so the map given by $g \mapsto \prod_{p \in V\mathcal{P}} g_p|_{\pi^{-1}(p)}$ is a homomorphism from $G_{(\mathcal{P})}$ to $\prod_{p \in V\mathcal{P}} G_{(\mathcal{P})}^p$. It is easily verified that this map is an isomorphism, from which it follows that G has property (P). \square

There are many mild conditions under which $\mathcal{U}_{\mathcal{L}}(M, N)$ leaves invariant no proper non-empty subtree of T and fixes non end of T . For example: M and N contain no fixed points.

We will presently cite a result in J. P. Serre's book [17], but we must do so with care since Serre's definition of a graph differs from ours in that loops (arcs a that satisfy $o(a) = t(a)$) and multiple arcs (distinct arcs a, a' that satisfy $o(a) = o(a')$ and $t(a) = t(a')$) are permitted. Trees in [17] cannot contain loops nor multiple arcs, and so we may use the term tree without ambiguity (see [17, pp. 13–17]).

A group G acts on a tree T *without inversion* if there is no edge $\{v, w\} \in T$ for which (v, w) lies in the orbit $G(w, v)$. Suppose T is a tree and G acts on T without inversion. Following [17, pp. 25], $G \backslash T$ consists of a set of vertices and a set of edges. The vertices (resp. edges) are the orbits of G on the vertices (resp. edges) of T . Notions like adjacency and arcs extend naturally to $G \backslash T$. Note that $G \backslash T$ may not be a graph (according to our definition), since there may be more than one edge between two given vertices.

Theorem 14 ([17, Corollary 1 in Section 5.4]). *Suppose T is a tree, and let G be a group acting on T without inversion. Let R be the group generated by all vertex stabilisers G_v , $v \in VT$. Then R is a normal subgroup of G , and G/R is isomorphic to the fundamental group of $G \backslash T$.* \square

It is an immediate consequence of this theorem that $G = R$ if and only if $G \backslash T$ is a tree (see [17, Exercise 2 in Section 5.4], for example).

As before, let X and Y be finite or infinite sets with cardinality at least two, $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$, denote the $(|X|, |Y|)$ -biregular tree by T and choose some legal colouring \mathcal{L} . Our next lemma follows from Proposition 4 and Lemma 10. In particular, we have that $\mathcal{U}_{\mathcal{L}}(M, N) \backslash T$ contains no loops or multiple edges.

Lemma 15. *If m (resp. n) denotes the number of orbits of M (resp. N), then $\mathcal{U}_{\mathcal{L}}(M, N) \backslash T$ is the complete bipartite graph $K_{m,n}$.* \square

If M and N are transitive, then Lemma 15 and [17, Section 4.1, Theorem 6] imply that $G := \mathcal{U}_{\mathcal{L}}(M, N)$ has an amalgamated free product structure,

$$G = G_u *_{G_{(u,w)}} G_v,$$

where u and v are any two adjacent vertices in T .

Recall that $M \boxtimes N$ is the subgroup of $\text{Sym}(V_Y)$ that is induced by $\mathcal{U}_{\mathcal{L}}(M, N)$. Following [11], we say a permutation group $G \leq \text{Sym}(V)$ is *generated by point stabilisers* if $G = \langle G_{\alpha} : \alpha \in V \rangle$.

Theorem 16. *Suppose M and N are permutation groups of (not necessarily finite) degree at least two, both groups are generated by point stabilisers and at least one group is nontrivial. Then $M \boxtimes N$ is simple if and only if M or N is transitive.*

Proof. Suppose M is a subgroup of $\text{Sym}(X)$ and N is a subgroup of $\text{Sym}(Y)$; let T and \mathcal{L} be as above. Write $G := \mathcal{U}_{\mathcal{L}}(M, N)$ and denote the stabiliser of $v \in VT$ in G^+ by G_v^+ .

We claim that $G_v^+ = G_v$ for all $v \in VT$. Given $v \in VT$, it follows from Lemma 7 that the group $G_v|_{B(v)}$ is generated by point stabilisers. Moreover, $G_{(v,w)}|_{B(v)} \leq G_v^+|_{B(v)}$ for all $w \in B(v)$. Hence $G_v^+|_{B(v)} = G_v|_{B(v)}$. In particular, G_v^+ and G_v both have the same orbits on $B(v)$. Since $G_{(v,w)} \leq G_v^+$ for all $w \in B(v)$, our claim follows.

Hence $R := \langle G_v : v \in VT \rangle \leq G^+$. Since M or N is nontrivial, R is nontrivial. Moreover, by Theorem 14, R is a normal subgroup of G , and $G = R$ if and only if $G \backslash T$ is a tree.

If M and N are intransitive, then $G \backslash T$ is not a tree by Lemma 15, and so R is a nontrivial proper normal subgroup of G . Conversely, suppose M or N is transitive. Then $G \backslash T$ is a tree, and so $G = R = G^+$.

Let ϵ be an end in T , and suppose T' is some non-empty proper subtree of T . Now $G_v|_{B(v)}$ is transitive for all v in one part of the bipartition of T , and so we may choose such a vertex v from $VT \setminus VT'$. Clearly G_v does not fix ϵ , nor does G_v leave T' invariant. Thus, by Theorem 13, $G = G^+$ is simple. Since $M \boxtimes N \cong G$, the result follows. \square

Since point stabilisers are maximal in primitive permutation groups, all non-regular primitive permutation groups are generated by any two distinct point stabilisers.

Corollary 17. *If M and N are non-regular primitive permutation groups, then $M \boxtimes N$ is simple.* \square

Corollary 18. *Under the conditions of Theorem 16, the group $\mathcal{U}_{\mathcal{L}}(M, N)$ is simple if and only if M or N is transitive.* \square

5. PERMUTATIONAL PROPERTIES

Recall the famous primitivity conditions for the unrestricted wreath product $M \text{ Wr}_Y N$ in its product action on the set X^Y of functions from Y to X (see [6, Lemma 2.7A] for example).

- (i) $M \text{ Wr}_Y N$ is transitive if and only if M is transitive; and
- (ii) $M \text{ Wr}_Y N$ is primitive if and only if M is primitive but not regular, and N is transitive and finite.

Because of this criteria, the wreath product can be used to easily build new primitive groups from other primitive groups. It features prominently in the seminal O’Nan–Scott Theorem, which classifies the finite primitive permutation groups (see [10], for example).

Compare the primitivity criteria for the wreath product above with the following astonishing result for the box product.

Theorem 19. *Given nontrivial permutation groups $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$, the permutation group $M \boxtimes N \leq \text{Sym}(V_Y)$ satisfies:*

- (i) $M \boxtimes N$ is transitive if and only if M is transitive; and
- (ii) $M \boxtimes N$ is primitive if and only if M is primitive but not regular, and N is transitive.

Despite the striking similarity between these two sets of conditions, $M \text{Wr}_Y N$ and $M \boxtimes N$ distort the actions of M and N in opposite ways. This is most apparent when M is subdegree-finite and primitive but not regular and N is subdegree-finite and transitive. All nontrivial orbital graphs of $M \boxtimes N$ are tree-like: they are locally finite, connected and have infinitely many ends; on the other hand nontrivial orbital graphs of $M \text{Wr}_Y N \leq \text{Sym}(X^Y)$ are locally finite and connected but with at most one-end (see [19, Theorem 2.4]). Figure 2 shows (left) an orbital graph of $S_3 \boxtimes S_2$ (which is necessarily infinite) and (right) an orbital graph of $S_3 \text{Wr} S_2$ (which is necessarily finite).

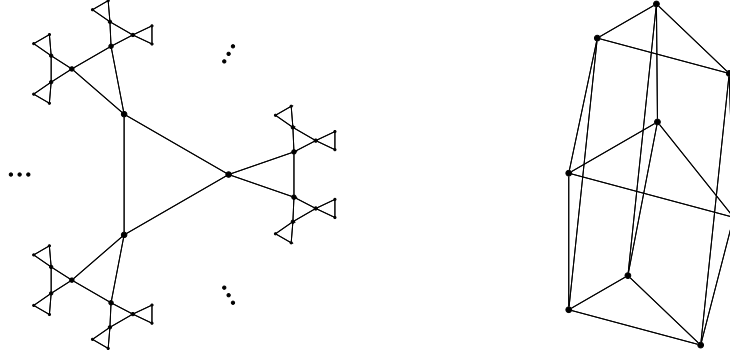


FIGURE 2. An orbital graph of $S_3 \boxtimes S_2$ (left) and of $S_3 \text{Wr} S_2$ (right)

Proof of Theorem 19. Part (i) is Theorem 2 (i). Write $G := \mathcal{U}_{\mathcal{L}}(M, N)$ for some legal colouring of X and Y , let T be the $(|X|, |Y|)$ -biregular tree.

We show first that if G is primitive on V_Y , then M is primitive but not regular and N is transitive. Suppose that N is not transitive on Y . Fix $v \in V_X$ and distinct vertices $w, w' \in B(v) \subseteq V_Y$. Choose $v' \in B(w)$ such that $\mathcal{L}(w, v)$ and $\mathcal{L}(w, v')$ lie in distinct orbits of N . By Proposition 4, the vertex v' does not lie in the orbit Gv . Let Γ be the orbital graph whose vertex set is V_Y and edge set is the orbit $G\{w, w'\}$. Notice that if two vertices in Γ are adjacent, then their distance in T is two. Therefore, if $x \in V_Y$ lies in the connected component of $T \setminus \{w\}$ that contains v' , then any path in Γ from x to w must contain an edge (in Γ) between w and some vertex $w'' \in B(v')$. But this implies that $\{w, w''\}$ lies in the orbit $G\{w, w'\}$, which requires that $v' \in Gv$, and we already have that $v' \notin Gv$. Hence G has a nontrivial orbital graph on V_Y that is not connected. The connected components of this graph give rise to a G -invariant equivalence relation on V_Y , and so the action of G on V_Y is not primitive.

Suppose M is not primitive on X . If M is not transitive then G is not transitive (and therefore not primitive) on V_Y by Proposition 4. Suppose then that M is imprimitive. By D. G. Higman's Theorem ([9, 1.12]), there exists an orbital graph of M that is nontrivial and not connected. Choose a vertex $v \in V_X$. By Lemma 7, some nontrivial orbital graph Δ of $G_v|_{B(v)}$ on $B(v)$ is not connected; let $\{w, w'\}$ be an edge in Δ . Now take Γ to be the orbital graph whose vertex set is V_Y and whose edge set is the orbit $G\{w, w'\}$. We claim that Γ is not connected. Indeed, any pair of adjacent vertices in Γ are at distance 2 in the tree T . Therefore any

Γ -path between distinct vertices in $B(v)$ contains only vertices in $B(v) = V\Delta$. It follows that Γ is not connected, and thus that G is not primitive on V_Y .

Suppose that M is regular. Let Γ be a new graph on V_Y , in which vertices are adjacent if and only if their distance in T is two. Thus Γ has connectivity one. If $|X| = 2$, then Γ is a regular tree and the bipartition of Γ is a system of imprimitivity for G on V_Y so the action of G on V_Y is not primitive. If $|X| \geq 3$, then the lobes of Γ contain at least three vertices, $G|_{V_Y} \leq \text{Aut } \Gamma$ acts vertex transitively, and the block-cut-vertex tree of Γ is T . Choose any vertex $v \in V_X$, and distinct vertices $w, w' \in B(v) \subseteq V_Y$. Because M is regular, $G_{w,v} = G_{w',v}$. By [18, Theorem 2.5], the action of G on V_Y is not primitive.

We have shown that if G is primitive on V_Y , then M is primitive but not regular and N is transitive; let us turn our attention to the converse. Suppose that M is primitive and not regular, and N is transitive. Let \sim be a nontrivial G -invariant equivalence relation on V_Y . Choose distinct $w, w' \in V_Y$ with $w \sim w'$. Let v denote the vertex adjacent to w' in the path $[w, w']_T$, and let w'' denote the vertex adjacent to v in $[w, w']_T$. Since $G_v|_{B(v)}$ is permutation isomorphic to M , it is primitive and not regular on $B(v)$, and so there is an element $h \in G_{v,w''}$ that does not fix w' . By Lemma 11, there exists $g \in G$ such that g fixes the half-tree $T_{(w'',v)}$ pointwise (so in particular g fixes w) and $gw' = hw'$. Hence $w' \sim w = gw \sim gw'$, and $d_T(w', gw') = 2$. However, $G_v|_{B(v)}$ is primitive, so \sim must be a universal relation on $B(v)$. Since N is transitive, it follows that \sim is universal on V_Y . Hence there are no proper nontrivial G -invariant equivalence relations on V_Y , so the action of G on V_Y is primitive. \square

Recall that a permutation group is *subdegree-finite* if any point stabiliser has only orbits of finite length.

Proposition 20. *Suppose M and N are nontrivial permutation groups. Then $M \boxtimes N$ is subdegree-finite if and only if M is subdegree-finite and all orbits of N are finite.*

Proof. Let $G := \mathcal{U}_{\mathcal{L}}(M, N)$. Suppose M is subdegree-finite and all orbits of N are finite. Choose distinct $w, w' \in V_Y$ and let $w_0 w_1 \cdots w_n$ denote the path $[w, w']_T$. Since $G_{w,w'} = G_{w_0, \dots, w_n}$ we have

$$|G_{w,w'}| = |G_{w_0} : G_{w_0, \dots, w_n}| \leq |G_{w_0} w_1| \cdot \prod_{i=1}^{n-1} |G_{w_{i-1}, w_i} w_{i+1}|.$$

Now $|G_{w_0} w_1|$ is finite because N has only finite orbits, and each $|G_{w_{i-1}, w_i} w_{i+1}|$ is finite because both M and N are subdegree-finite. Hence the action of G on V_Y is subdegree-finite.

On the other hand, if M is not subdegree-finite, then for all $v \in V_X$ there exist $w, w' \in B(v) \subseteq V_Y$ such that $G_{w,v} w'$ is infinite. Hence $G_w w'$ is not finite. If N has an infinite orbit, then for all $w \in V_Y$ there exists $v \in B(w) \subseteq V_X$ such that $G_w v$ is infinite. Hence for any $w' \in B(v) \setminus \{w\}$, the orbit $G_w w'$ is infinite. \square

6. TOPOLOGICAL PROPERTIES OF THE PRODUCT

If we bestow $\text{Sym}(X)$ and $\text{Sym}(Y)$ with their respective permutation topologies, then $M \boxtimes N$ under the permutation topology of $\text{Sym}(V_Y)$ preserves some topological properties of M and N but does not preserve discreteness. In this section, all topological statements are with respect to the permutation topology, which is described in Section 2.3.

Theorem 21. *Suppose $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$ are nontrivial and closed. Then $M \boxtimes N$ is closed in $\text{Sym}(V_Y)$ and the following are equivalent:*

- (i) *every point stabiliser in $M \boxtimes N$ is compact in $\text{Sym}(V_Y)$;*
- (ii) *N is compact and every point stabiliser in M is compact.*

This theorem is an immediate consequence of Lemma 8 and the following lemma.

Lemma 22. *Let T be a tree with no vertex of valence one. Suppose $G \leq \text{Aut } T$ is closed and fixes setwise the two parts V_1, V_2 of the bipartition of T . Then $G|_{V_i}$ is closed in $\text{Sym}(V_i)$ for $i = 1, 2$, and if $G_v|_{B(v)}$ is closed for all $v \in VT$ then the following are equivalent:*

- (i) *for all $v \in V_1$ and $w \in V_2$, all point stabilisers in $G_v|_{B(v)}$ are compact and $G_w|_{B(w)}$ is compact;*
- (ii) *all point stabilisers in $G|_{V_2}$ are compact.*

Proof. Let $A := (\text{Aut } T)_{\{V_1\}}$ and let $\varphi : A \rightarrow A|_{V_1}$ be the map taking $g \in A$ to $g|_{V_1} \in A|_{V_1}$. Since the tree T contains no vertices of valency one, the kernel of this map is trivial and φ is an isomorphism. Now A is clearly closed, and $\varphi(A)$ is closed in $\text{Sym}(V_1)$ since it is the full automorphism group of the graph Γ whose vertex set is V_1 and whose edge set is defined by the rule that two vertices are adjacent in Γ if and only if their distance in T is 2.

For any finite set $\Phi \subseteq VT$, there exists a finite set $\Phi' \subseteq V_1$ such that $A_{(\Phi')} \leq A_{(\Phi)}$. Furthermore, $(\varphi(A))_{(\Phi')} = \varphi(A_{(\Phi')}) \leq \varphi(A_{(\Phi)})$, so $\varphi(A_{(\Phi)})$ is open in $\varphi(A)$. An open set in A is a union of cosets of pointwise stabilisers of finite subsets of VT , so it follows that the image under φ of any open set in A is open in $\varphi(A)$. By assumption $A \setminus G$ is open in A , so $\varphi(A) \setminus \varphi(G) = \varphi(A \setminus G)$ is open in $\varphi(A)$ and hence $\varphi(G)$ is closed in $\varphi(A)$. Since $\varphi(A)$ is closed in $\text{Sym}(V_1)$, it follows that $G|_{V_1}$ is closed in $\text{Sym}(V_1)$. A symmetric argument shows that $G|_{V_2}$ is closed in $\text{Sym}(V_2)$.

Now suppose $G_v|_{B(v)}$ is closed for all $v \in VT$. Note that (i) is true if and only if for all $v \in V_1$ and $w \in V_2$ we have that all orbits of $G_w|_{B(w)}$ are finite and all suborbits of $G_v|_{B(v)}$ are finite. On the other hand (ii) is true if and only if $G|_{V_2}$ is subdegree-finite. An argument similar to that used in the proof of Proposition 20 can therefore be used to show that (i) and (ii) are equivalent. \square

Theorem 23. *Suppose $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$ are transitive. If, in their respective permutation topologies,*

- (i) *M is compactly generated and every point stabiliser in M is compact; and*
- (ii) *N is compact,*

then $M \boxtimes N$ is compactly generated and every point stabiliser in $M \boxtimes N$ is compact.

Proof. Suppose (i) and (ii) are true. Let \mathcal{L} be a legal colouring of the $(|X|, |Y|)$ -biregular tree T and write $G := \mathcal{U}_{\mathcal{L}}(M, N)$. Fix $v \in V_X$ and a neighbour $w \in B(v) \subseteq V_Y$. Let $H := G_v|_{B(v)}$. Since H is permutation isomorphic to M , we may choose some finite set $S' := \{s'_1, \dots, s'_n\}$ such that $H = \langle H_w, s'_1, \dots, s'_n \rangle$.

For each s'_i , choose some $s_i \in G_v$ such that $s_i|_{B(v)} = s'_i$, and write $S := \{s_1, \dots, s_n\}$. Consider the group $K := \langle G_w \cup S \rangle$. Since G_w is transitive on $B(w)$, the stabiliser K_w is transitive on $B(w)$.

We claim that K_v is transitive on $B(v)$. Indeed, $S \subseteq K_v$, and $G_{v,w} \leq K_v$, with $G_{v,w}|_{B(v)} = H_w$ and $S|_{B(v)} = S'$. Since $H = \langle H_w \cup S' \rangle$ is transitive on $B(v)$, our claim follows.

Since K_w is transitive on $B(w)$ and K_v is transitive on $B(v)$, the group K has two orbits, V_X and V_Y , on VT . Thus $G_w < K \leq G$ and the orbits Kw and Gw are equal. Hence K and G must also be equal.

We have shown that $M \boxtimes_{\mathcal{L}} N = G|_{V_Y} = \langle G_w|_{V_Y} \cup S|_{V_Y} \rangle$. Now N is a compact subgroup of the Hausdorff group $\text{Sym}(Y)$, so N is closed. Similarly, every point stabiliser in M is closed, so M is also closed. Therefore, by Lemma 22, every point stabiliser in $G|_{V_Y}$ is compact. Hence $G_w|_{V_Y} \cup S|_{V_Y}$ is a compact generating set for $G|_{V_Y}$. \square

Recall that a permutation group is *semi-regular* if every point stabiliser is trivial.

Theorem 24. *Suppose $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$. Then $M \boxtimes N$ is discrete if and only if M and N are semi-regular.*

Proof. Let \mathcal{L} be a legal colouring of the $(|X|, |Y|)$ -biregular tree T and write $G := \mathcal{U}_{\mathcal{L}}(M, N)$. Suppose M and N are both semi-regular. If $w, w' \in V_Y$ are at distance two in T , there is a unique element $v \in V_X$ that is adjacent to both vertices, and $G_{w,w'} = G_{w,v,w'}$. By Lemma 7, $G_{w,v}$ fixes $B(w)$ and $B(v)$ pointwise, and, since T is connected, $G_{w,v}$ must therefore fix VT pointwise; it is thus trivial. Hence $(M \boxtimes N)_{w,w'}$ is trivial and $M \boxtimes N$ is discrete in $\text{Sym}(V_Y)$.

Let $\Phi \subseteq V_Y$ be any finite set. Suppose M is not semi-regular, and choose $x \in X$ such that M_x is nontrivial. For each $w' \in VT$, the connected components of $T \setminus B(w')$ each contain infinitely many vertices $w \in V_Y$ such that $\mathcal{L}|_{\overline{A}(w)} = x$, so we may choose a pair of adjacent vertices $v \in V_X, w \in V_Y$ such that $\mathcal{L}(v, w) = x$ and Φ is contained in the half-tree $T_{(w,v)}$. Choose $\sigma \in M_x \setminus \langle 1 \rangle$. By Lemma 7, there exists $h \in G_v$ such that $\mathcal{L}|_{A(v)} h|_{A(v)} \mathcal{L}|_{A(v)}^{-1} = \sigma$. Hence $h \in G_{v,w}$ and $h \notin G_{B(v)}$. By Lemma 11, there exists $g \in G_{v,w}$ such that $g|_{T_{(w,v)}}$ is trivial and $g|_{T_{(v,w)}} = h|_{T_{(v,w)}}$. Hence $g \in G_{(\Phi)}$ is nontrivial. A similar argument shows that if N is not semi-regular, then $G_{(\Phi)}$ is nontrivial. It follows that if M or N is not semi-regular, then $M \boxtimes N$ is not discrete. \square

Corollary 25. *Suppose X and Y are countable, and $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$ are closed. Then*

$$|M \boxtimes N| \leq \aleph_0 \quad \text{or} \quad |M \boxtimes N| = 2^{\aleph_0},$$

with the former holding if and only if M and N are semi-regular.

Proof. By Theorem 21, the group $G := M \boxtimes N$ is a closed group of permutations of the countably infinite set V_Y . Since the trivial group $\langle 1 \rangle$ is closed, we may deduce from a theorem of D. M. Evans ([7, Theorem 1.1]) that either $|G| = 2^{\aleph_0}$ or the pointwise stabiliser (in G) of some finite subset of V_Y is trivial. But the pointwise stabiliser of some finite subset of V_Y is trivial if and only if G is discrete. The corollary now follows from Theorem 24. \square

7. AN UNCOUNTABLE SET OF COMPACTLY GENERATED SIMPLE GROUPS

In [4], \mathcal{S} is defined to be the set of non-discrete, compactly generated, topologically simple, totally disconnected, locally compact groups. P.-E. Caprace and T. De Medts remark that they do not know whether \mathcal{S} contains uncountably many pairwise non-isomorphic compactly generated groups. In this section we prove that \mathcal{S} is uncountable, by constructing the first uncountable set of pairwise non-isomorphic simple groups that are non-discrete, compactly generated, totally disconnected and locally compact. In fact, we construct 2^{\aleph_0} such groups.

In [5] the known examples of groups in \mathcal{S} are organised into five broad classes. The examples, of which there are countably many, that are structurally similar to

the groups we describe here are due to M. Burger and S. Mozes ([3]), R. G. Möller and J. Vonk ([11]), and C. Banks, M. Elder and G. A. Willis ([1]).

Recall from [17, pp. 58] the definition of Serre's property (FA). Suppose G is a group acting on a tree T without inversion. Let $\text{Fix}_G(T)$ be the set of vertices of T that are fixed by all elements in G . A group G has property (FA) if $\text{Fix}_G(T)$ is non-empty for any tree T on which G acts without inversion.

For denumerable groups, having property (FA) is equivalent to being finitely generated but not an amalgam, with no quotient isomorphic to \mathbb{Z} (see [17, Theorem 15]). A particularly rich class of groups (for our purposes) that all have property (FA) is the class of finitely generated torsion groups.

Proposition 26 ([17, Example 6.3.1]). *A finitely generated torsion group has property (FA).* \square

Theorem 27 ([15, Theorem 28.7]). *For every sufficiently large prime number p , there is a continuum of pairwise non-isomorphic infinite groups of exponent p all of whose proper nontrivial subgroups have order p .* \square

In [15, pp. 304] Ol'Shanskiĭ notes that taking $p > 10^{75}$ in the above theorem is sufficient; let us now fix such a prime p . The groups whose existence is guaranteed by Theorem 27 we shall call *Taski-Ol'Shanskiĭ Monsters*.

Let Q be a Taski-Ol'Shanskiĭ Monster. It is 2-generated and torsion, so by Proposition 26 it has property (FA). It is easily seen to be simple, since if N is a proper normal subgroup of Q , then Q/N is infinite and torsion so it contains a nontrivial finite subgroup H/N , with $N < H < Q$.

Lemma 28. *Let M_1, M_2, N_1, N_2 be nontrivial permutation groups. Suppose M_1 has property (FA), and no nontrivial quotient of M_1 is isomorphic to any subgroup of M_2 or N_2 . Then $M_1 \boxtimes N_1$ and $M_2 \boxtimes N_2$ are not (abstractly) isomorphic.*

Proof. For $i = 1, 2$, suppose $M_i \leq \text{Sym}(X_i)$ and $N_i \leq \text{Sym}(Y_i)$ and let T_i denote the $(|X_i|, |Y_i|)$ -biregular tree. Let \mathcal{L}_i be a legal colouring of X_i and Y_i , and write $G := \mathcal{U}_{\mathcal{L}_1}(M_1, N_1)$ and $H := \mathcal{U}_{\mathcal{L}_2}(M_2, N_2)$. It suffices to show that G is not isomorphic to H . Let us suppose, for a contradiction, that G and H are isomorphic. By Proposition 9, G contains a subgroup that is isomorphic to M_1 , and so M_1 is isomorphic to some subgroup K of H .

On one hand, K cannot fix any vertex in T_2 . Indeed, if K fixes some vertex $v \in T_2$, then $K/K_{(B(v))} \cong K|_{B(v)} \leq H_v|_{B(v)}$. Since $H_v|_{B(v)}$ is isomorphic to either M_2 or N_2 , this is only possible if the quotient $K/K_{(B(v))}$ is trivial; that is, if K fixes $B(v)$ pointwise. But if this is so, then we can repeat this argument, since K now fixes $w \in B(v)$. Since T_2 is connected, it follows that K must be trivial, which is absurd.

On the other hand, K must fix a vertex in T_2 , since K acts on T_2 without inversion and has property (FA). \square

Theorem 29. *There are uncountably many non-isomorphic groups that are totally disconnected, locally compact, compactly generated, simple and not discrete.*

Proof. Let Q be a Taski-Ol'Shanskiĭ Monster, and let S_3 denote the symmetric group of $Y := \{1, 2, 3\}$. Fix a nontrivial proper subgroup $H \leq Q$. The group Q is simple and it acts faithfully and transitively on the coset space $X := (Q : H)$, so we think of Q as being a subgroup of $\text{Sym}(X)$, and bestow upon it the permutation topology. Point stabilisers in Q are finite, so Q is totally disconnected and locally compact, with compact stabilisers. Moreover, Q is finitely generated, so it is compactly generated.

Let T be the $(\aleph_0, 3)$ -biregular tree, and bipartition the vertices of T into sets V_X and V_Y in the usual way. The group $Q \boxtimes S_3$ is a subgroup of $\text{Sym}(V_Y)$. Under the permutation topology, $Q \boxtimes S_3$ is totally disconnected because its action is faithful. It is locally compact by Theorem 21, it is compactly generated by Theorem 23, and it is non-discrete by Theorem 24.

Any point stabiliser in Q is a maximal subgroup, so Q is generated by point stabilisers; so too is S_3 . Hence, by Theorem 16, the group $Q \boxtimes S_3$ is simple.

If Q' is another Taski-Ol'Shanskiĭ Monster that is not isomorphic to Q , then $Q \boxtimes S_3$ and $Q' \boxtimes S_3$ are not isomorphic by Lemma 28. The theorem now follows immediately from Theorem 27. \square

Remark 30. Of course, the construction described above works for many groups that are not Taski-Ol'Shanskiĭ Monsters. The following is a general method for constructing examples of non-discrete simple groups that are totally disconnected, locally compact, and compactly generated.

- (i) Choose $M \leq \text{Sym}(X)$ to be transitive and non-regular, with all point stabilisers compact and open, such that M is compactly generated and generated by point stabilisers. Note that we do not require M to be non-discrete.

Example: Take any locally finite, vertex-transitive, connected graph Γ that contains at least two vertices, set $X := V\Gamma$ and choose a closed vertex-transitive group $M \leq \text{Aut } \Gamma$ that is generated by vertex-stabilisers. In the permutation topology of $\text{Sym}(V\Gamma)$, such a group will be totally disconnected and locally compact, and all vertex stabilisers will be compact and open, and by a result of R. G. Möller ([12, Corollary 1]) the group M is compactly generated.

Example: Take $M \leq \text{Sym}(X)$ to be a closed subdegree-finite non-regular primitive permutation group of degree at least three.

Example: Take an abstract group H that contains a nontrivial proper finite subgroup K such that H is generated by finitely many H -conjugates of K . Let X be the set of left cosets of K in H . If the kernel of the natural action of H on X is not equal to K , then take $M \leq \text{Sym}(X)$ to be the permutation group induced by this action.

- (ii) Choose any finite transitive permutation group $N \leq \text{Sym}(Y)$ of degree at least two, that is generated by point stabilisers. For example, one could take N be any finite primitive non-regular permutation group of degree at least three. In the permutation topology, such a group will be totally disconnected and compact.
- (iii) Choose any legal colouring \mathcal{L} of the $(|X|, |Y|)$ -biregular tree T , and let V_Y be the part of the bipartition of T whose elements have valence $|Y|$.
- (iv) The group $M \boxtimes N \leq \text{Sym}(V_Y)$ is simple (by Theorem 16), and with the permutation topology of $\text{Sym}(V_Y)$ it is totally disconnected (because it is faithful), locally compact (by Theorem 21), compactly generated (by Theorem 23) and non-discrete (by Theorem 24). Point stabilisers in $M \boxtimes N$ are compact.

Remark 31. In [21, Problem 4.3], G. A. Willis asks the following. Let G_1, G_2 be (non-discrete) topologically simple (or simple) totally disconnected locally compact groups. Suppose that there are compact open subgroups $U_i \leq G_i$ ($i = 1, 2$) that are isomorphic. Does it follow that G_1 and G_2 are isomorphic? It is known (see [2] and [4]) that the answer to this question is no. However, only countably many pairs (G_1, G_2) demonstrating this have been found. Using the box product, it is easy to construct an uncountable set of such groups $\{G_i : i \in I\}$, in which each group G_i

contains a compact open subgroup $U_i \leq G_i$ such that for all $i, j \in I$ the groups U_i and U_j are isomorphic as topological groups but G_i and G_j are non-isomorphic.

Indeed, for any two groups $G := Q \boxtimes S_3$ and $H := Q' \boxtimes S_3$ taken from the proof of Theorem 29, any point stabiliser G_v in G is permutation isomorphic to any point stabiliser in H . This is easy to see if you consider the induced actions of G and H on the $(\aleph_0, 3)$ -biregular tree T .

Thus, if Q and Q' are non-isomorphic, then G and H are non-isomorphic, non-discrete, simple, totally disconnected locally compact groups and moreover G and H have compact open subgroups, G_v and H_v respectively, which are isomorphic as topological groups.

Remark 32. The 2^{\aleph_0} groups of the form $Q \boxtimes S_3$ described in the proof of Theorem 29 are non-discrete, infinite and primitive (as permutation groups). Any distinct pair G, H of such groups has the property that every point stabiliser in G is permutation isomorphic to every point stabiliser in H (again, this is easy to see if you consider the induced actions of G and H on the $(\aleph_0, 3)$ -biregular tree T). Nevertheless, G and H are not isomorphic.

Acknowledgments The author would like to thank R. G. Möller for providing him with an English translation of Tits' paper ([20]).

REFERENCES

- [1] C. Banks, M. Elder, G. A. Willis, Simple groups of automorphisms of trees determined by their actions on finite subtrees, Preprint (2013), arXiv:1312.2311
- [2] Y. Barnea, M. Ershov, T. Weigel, Abstract commensurators of profinite groups, Trans. Amer. Math. Soc. 363 (2011) 5381–5417.
- [3] M. Burger and S. Mozes, Groups acting on trees: from local to global structure, Publications mathématiques de l'I.H.É.S. tome 92 (2000) 113–150.
- [4] P.-E. Caprace and T. De Medts, Simple locally compact groups acting on trees and their germs of automorphisms, Transformation Groups, 16 (2011) 375–411.
- [5] P.-E. Caprace, C. D. Reid, G. A. Willis, Locally normal subgroups of totally disconnected groups. Part II: Compactly generated simple groups, Preprint (2013) arXiv:1401.3142
- [6] J. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics 163 (Springer-Verlag, New York, 1996).
- [7] D. M. Evans, A note on automorphism groups of countably infinite structures, Arch. Math. 49 (1987) 479–483.
- [8] D. M. Evans, An infinite highly arc-transitive digraph, Europ. J. Combinatorics 18 (1997) 281–286.
- [9] D. G. Higman, Intersection matrices for finite permutation groups, J. Algebra 6 (1967) 22–42.
- [10] M. W. Liebeck, C. E. Praeger, and J. Saxl, 'On the O'Nan–Scott Theorem for finite primitive permutation groups', J. Austral. Math. Soc. 44 (1988) 389–396.
- [11] R. G. Möller and J. Vonk, Normal subgroups of groups acting on trees and automorphism groups of graphs, J. Group Th. 15 (2012) 831–850.
- [12] R. G. Möller, FC^- -elements in totally disconnected groups and automorphisms of infinite graphs, Math. Scand. 92 (2003) 261–268.
- [13] R. G. Möller, Structure theory of totally disconnected locally compact groups via graphs and permutations, Canad. J. Math. 54 (2002) 795–827.
- [14] V. N. Obratsov, Embedding into groups with well-described lattices of subgroups, Bull. Aust. Math. Soc. 54 (1996) 221–240.
- [15] A. Yu. Ol'Shanskiĭ, *Geometry of defining relations in groups*, Mathematics and its Applications (Soviet Series) 70 (Kluwer Acad. Publ., Dordrecht, 1991).
- [16] D. Osin, Small cancellations over relatively hyperbolic groups and embedding theorems, Ann. of Math. 172 (2010) 1–39.
- [17] J.-P. Serre, *Trees*, Springer-Verlag, 2003.
- [18] S. M. Smith Infinite primitive directed graphs, J. Algebr. Comb. 31 (2010) 131–141.
- [19] S. M. Smith, Subdegree growth rates of infinite primitive permutation groups, J. London Math. Soc. 82 (2010) 526–548; doi: 10.1112/jlms/jdq046.

- [20] J. Tits, Sur le groupe des automorphismes d'un arbre in Essays on Topology and Related Topics: Mémoires dédiés à Georges de Rham, A. Haefliger, R. Narasimhan (eds), Berlin and New York, Springer-Verlag (1970) 188–211.
- [21] G. A. Willis, Compact open subgroups in simple totally disconnected groups, *J. Algebra* 312 (2007) 405–417.
- [22] W. Woess, 'Topological groups and infinite graphs', *Discrete Math.* 95 (1991) 373–384.

DEPARTMENT OF MATHEMATICS, NYC COLLEGE OF TECHNOLOGY, CITY UNIVERSITY OF
NEW YORK, NEW YORK, NY, USA

E-mail address: `sismith@citytech.cuny.edu`